A REMARK ON THE PERMUTATION REPRESENTATIONS AFFORDED BY THE EMBEDDINGS OF $\mathrm{O}_{2m}^{\pm}(2^f)$ IN $\mathrm{Sp}_{2m}(2^f)$

SIMON GUEST, ANDREA PREVITALI, AND PABLO SPIGA

Abstract. We show that the permutation module over $\mathbb{C}$ afforded by the action of $\mathrm{Sp}_{2m}(2^f)$ on its natural module is isomorphic to the permutation module over $\mathbb{C}$ afforded by the action of $\mathrm{Sp}_{2m}(2^f)$ on the union of the right cosets of $\mathrm{O}_{2m}^{+}(2^f)$ and $\mathrm{O}_{2m}^{-}(2^f)$.

1. Introduction

That a given finite group can have rather different permutation representations affording the same permutation character was shown by Helmut Wielandt in 1979. For instance, the actions of the projective general linear group $\mathrm{PGL}_d(q)$ on the projective points and on the projective hyperplanes afford the same permutation character, but these actions are not equivalent when $d \geq 3$. A more interesting example is offered by the Mathieu group M_{23}. Here we have two primitive permutation representations of degree 253 affording the same permutation character, but with non-isomorphic point stabilizers (see [3, p. 71]).

Establishing which properties are shared by permutation representations of a finite group G with the same permutation character has been the subject of considerable interest. For instance, it was conjectured by Wielandt [15] that, if G admits two permutation representations on Ω_1 and Ω_2 that afford the same permutation character, and if G acts primitively on Ω_1, then G acts primitively on Ω_2. This conjecture was first reduced to the case that G is almost simple by Förster and Kovács [5] and then it was solved (in the negative) by Guralnick and Saxl [7]. Some more recent investigations on primitive permutation representations and their permutation characters can be found in [14].

In this paper we construct two considerably different permutation representations of the symplectic group that afford the same permutation character. We let q be a power of 2, G be the finite symplectic group $\mathrm{Sp}_{2m}(q)$, V be the $2m$-dimensional natural module for $\mathrm{Sp}_{2m}(q)$ over the field $\mathbb{F}_q$ of q elements, and π be the complex permutation character for the action (by matrix multiplication) of G on V. Since q is even, the orthogonal groups $\mathrm{O}_{2m}^{+}(q)$ and $\mathrm{O}_{2m}^{-}(q)$ are maximal subgroups of G (see [4]). For $\varepsilon \in \{+,-\}$, we let Ω^{ε} denote the set of right cosets of $\mathrm{O}_{2m}^{\varepsilon}(q)$ in G, and we let π^{ε} denote the permutation character for the action of G on Ω^{ε}.

Theorem 1. *The $\mathbb{C}G$-modules $\mathbb{C}V$ and $\mathbb{C}\Omega^{+} \oplus \mathbb{C}\Omega^{-}$ are isomorphic. That is,* $\pi = \pi^{+} + \pi^{-}$.

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Address correspondence to P. Spiga, E-mail: pablo.spiga@unimib.it.

We find this behaviour quite peculiar considering that the G-sets V and $\Omega^+ \cup \Omega^-$ are rather different. For instance, G has two orbits of size 1 and $q^{2m}-1$ on V, and has two orbits of size $q^m(q^m+1)/2$ and $q^m(q^m-1)/2$ on $\Omega^+ \cup \Omega^-$. Moreover, the action of G on both Ω^+ and Ω^- is primitive, but the action of G on $V \setminus \{0\}$ is not when $q > 2$.

2. Proof of Theorem 1

Inglis [8, Theorem 1] shows that the orbitals of the two orthogonal subgroups are self-paired, hence the characters π^+ and π^- are multiplicity-free (see [1, §2.7]). We will use this fact in our proof of Theorem 1.

Proof of Theorem 1. Let 1 denote the principal character of G. Observe that $\pi = 1 + \pi_0$, where π_0 is the permutation character for the transitive action of G on $V \setminus \{0\}$. In particular, for $v \in V \setminus \{0\}$, we have $\pi_0 = 1_{G_v}^G$, where G_v is the stabilizer of v in G. Frobenius reciprocity implies that $\langle \pi_0, \pi_0 \rangle = \langle \pi_0|_{G_v}, 1 \rangle$, and this equals the number of orbits of G_v on $V \setminus \{0\}$. We claim that G_v has $2q-1$ orbits on $V \setminus \{0\}$. More precisely we show that, given $w \in v^\perp \setminus \langle v \rangle$ and $w' \in V \setminus v^\perp$, the elements λv (for $\lambda \in \mathbb{F}_q \setminus \{0\}$), w, and $\lambda w'$ (for $\lambda \in \mathbb{F}_q \setminus \{0\}$) are representatives for the orbits of G_v on $V \setminus \{0\}$. Since G_v fixes v and preserves the bilinear form $(\ ,\ )$, these elements are in distinct G_v-orbits. Let $u \in V \setminus \{0\}$. If $u \in \langle v \rangle$, then $u = \lambda v$ for some $\lambda \neq 0$, and hence there is nothing to prove. Let $w_0 = w$ if $(v, u) = 0$, and let $w_0 = \frac{(v,u)}{(v,w')} w'$ if $(v, u) \neq 0$. By construction, the 2-spaces $\langle v, u \rangle$ and $\langle v, w_0 \rangle$ are isometric and they admit an isometry f such that $v^f = v$ and $u^f = w_0$. By Witt's Lemma [9, Proposition 2.1.6], f extends to an isometry g of V. Thus $g \in G_v$ and $u^g = w_0$, which proves our claim. Therefore, we have

$$\langle \pi_0, \pi_0 \rangle = 2q - 1. \tag{1}$$

Next we need to refine the information in (1). Let P be the stabilizer of the 1-subspace $\langle v \rangle$ in G. Then P is a maximal parabolic subgroup of G and P/G_v is cyclic of order $q-1$. Write $\eta = 1_{G_v}^P$ and observe that $\eta = \sum_{\zeta \in \mathrm{Irr}(P/G_v)} \zeta$, where by abuse of terminology we identify the characters of P/G_v with the characters of P containing G_v in the kernel. Thus

$$\pi_0 = 1_{G_v}^G = (1_{G_v}^P)_P^G = \eta_P^G = \sum_{\zeta \in \mathrm{Irr}(P/G_v)} \zeta_P^G.$$

Since every character of G is real-valued [6], we must have

$$(\overline{\zeta})_P^G = \overline{\zeta_P^G} = \zeta_P^G,$$

where $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$. Let $\mathcal{S}$ be a set of representatives, up-to-complex conjugation, of the non-trivial characters of $\mathrm{Irr}(P/G_v)$. Since $|P/G_v| = q-1$ is odd, we see that $|\mathcal{S}| = q/2 - 1$. We have

$$\pi_0 = 1_P^G + 2 \sum_{\zeta \in \mathcal{S}} \zeta_P^G.$$

If we write $\pi' = \sum_{\zeta \in \mathcal{S}} \zeta_P^G$, then we have $\pi_0 = 1_P^G + 2\pi'$.

Since 1_P^G is the permutation character of the rank 3 action of G on the 1-dimensional subspaces of V, we have $1_P^G = 1 + \chi^+ + \chi^-$ for some distinct non-trivial irreducible characters χ^+ and χ^- of G. Let Γ be the graph with vertex set

the 1-subspaces of V and edge sets $\{\langle v\rangle, \langle w\rangle\}$ whenever $v \perp w$. Observe that Γ is strongly regular with parameters

$$\left(\frac{q^{2m}-1}{q-1}, \frac{q^{2m-1}-q}{q-1}, \frac{q^{2m-2}-1}{q-1}-2, \frac{q^{2m-2}-1}{q-1}\right).$$

Hence the eigenvalues of Γ have multiplicity $\frac{1}{2}\left(\frac{q^{2m}-q}{q-1}-q^m\right)$ and $\frac{1}{2}\left(\frac{q^{2m}-q}{q-1}+q^m\right)$ (see [2, p. 27]).

Interchanging the roles of χ^+ and χ^- if necessary, we may assume that $\chi^-(1) < \chi^+(1)$. The above direct computation proves that

$$\chi^-(1) = \frac{1}{2}\left(\frac{q^{2m}-q}{q-1}-q^m\right) \quad \text{and} \quad \chi^+(1) = \frac{1}{2}\left(\frac{q^{2m}-q}{q-1}+q^m\right) \tag{2}$$

(compare [10, Section 1]).

Fix $\zeta \in \mathcal{S}$. We claim that ζ_P^G is irreducible. From Mackey's irreducibility Criterion [12, Proposition 23, Section 7.3], we need to show that for every $s \in G \setminus P$, we have $\zeta_{sPs^{-1}\cap P} \neq \zeta^s$, where ζ^s is the character of $sPs^{-1}\cap P$ defined by $(\zeta^s)(x) = \zeta(s^{-1}xs)$ and, as usual, $\zeta_{sPs^{-1}\cap P}$ is the restriction of ζ to $sPs^{-1}\cap P$. Fix a monomorphism ψ from P/P' into $\mathbb{C}^*$. Since ζ is a class function of P, we need to consider only elements s in distinct (P,P)-double cosets. These correspond to the P-orbits $\langle v\rangle$, $v^\perp \setminus \langle v\rangle$ and $V \setminus v^\perp$. Let $H = \langle v, u\rangle$ be a hyperbolic plane and choose $s \in G$ such that

$$vs = u, \quad us = u \quad \text{and} \quad s_{H^\perp} = 1_{H^\perp}.$$

A calculation shows that $\zeta^s(x) = \psi(\mu^{-1}) = \overline{\zeta(x)}$, where $vx = \mu v$. Since $q-1$ is odd, we have $\zeta(x) \neq \zeta^s(x)$ when $\mu \neq 1$. Therefore $\zeta \neq \zeta^s$. Finally choose $s \in G$ such that $(v,u,w,z)s = (w,z,v,u)$, where $H = \langle v,u\rangle \perp \langle w,z\rangle$ is an orthogonal sum of hyperbolic planes and $s_{H^\perp} = 1_{H^\perp}$. Another calculation shows that $\zeta^s(x) = \psi(\lambda)$ and $\zeta(x) = \psi(\mu)$, where $vx = \mu v$ and $wx = \lambda$. If $\mu \neq \lambda$, then $\zeta^s(x) \neq \zeta(x)$ and hence $\zeta^s \neq \zeta$. Our claim is now proved.

Write $\pi' = \sum_{i=1}^{\ell} m_i\chi_i$ as a linear combination of the distinct irreducible constituents of π'. Observe that, by the previous paragraph, each χ_i is of the form ζ_P^G, for some $\zeta \in \mathcal{S}$. Therefore χ_i has degree $|G:P|$ for each i and, in particular, 1, χ^+ and χ^- are not irreducible constituents of π'.

The number of irreducible constituents of π_0 is

$$1+1+1+2(m_1+\cdots+m_\ell) = 3+2|\mathcal{S}| = 3+2\left(\frac{q}{2}-1\right) = q+1,$$

and by (1) we have

$$3+4m_1^2+\cdots+4m_\ell^2 = 2q-1.$$

Multiplying the first equation by -2 and adding the second equation we have

$$-3+4m_1(m_1-1)+\cdots+4m_\ell(m_\ell-1) = -3.$$

It follows that $m_1 = \cdots = m_\ell = 1$, and hence $\ell = q/2-1$. This shows that π' is multiplicity-free.

Summing up, we have

$$\pi_0 = 1+\chi^++\chi^-+2\pi', \quad \langle\pi',\pi'\rangle = \frac{q}{2}-1, \quad \langle 1+\chi^++\chi^-,\pi'\rangle = 0. \tag{3}$$

We now turn our attention to the characters π^+ and π^-. By Frobenius reciprocity, or by [8, Theorem 1 (i) and (ii)], we see that

$$\langle \pi^+, \pi^+ \rangle = \langle \pi^-, \pi^- \rangle = \frac{q}{2} + 1. \tag{4}$$

By [8, Lemma 2 (iii) and (iv)], the orbits of $\mathrm{O}_{2m}^-(q)$ in its action on Ω^+ are in one-to-one correspondence with the elements in $\{\alpha + \alpha^2 \mid \alpha \in \mathbb{F}_q\}$. In particular, we have $\langle \pi^+|_{\mathrm{O}_{2m}^-(q)}, 1 \rangle = |\{\alpha + \alpha^2 \mid \alpha \in \mathbb{F}_q\}| = q/2$. Now Frobenius reciprocity implies that

$$\langle \pi^+, \pi^- \rangle = \frac{q}{2}. \tag{5}$$

Next we show that

$$\langle \pi_0, \pi^+ \rangle = \langle \pi_0, \pi^- \rangle = q. \tag{6}$$

Using Frobenius reciprocity, it suffices to show that the number of orbits of $\mathrm{O}_{2m}^\pm(q)$ on $V \setminus \{0\}$ is q. Fix $\varepsilon \in \{+, -\}$ and let Q^ε be the quadratic form on V preserved by $\mathrm{O}_{2m}^\varepsilon(q)$. For $\lambda \in \mathbb{F}_q$, we see from [9, Lemma 2.10.5 (ii)] that $\Omega_{2m}^\varepsilon(q)$ is transitive on $V_\lambda^\varepsilon = \{v \in V \setminus \{0\} \mid Q^\varepsilon(v) = \lambda\}$. In particular, $\{V_\lambda^\varepsilon \mid \lambda \in \mathbb{F}_q\}$ is the set of orbits of $\Omega_{2m}^\varepsilon(q)$ on $V \setminus \{0\}$. Since $\mathrm{O}_{2m}^\varepsilon(q)$ is the isometry group of Q^ε we see that $\{V_\lambda^\epsilon \mid \lambda \in \mathbb{F}_q\}$ is also the set of orbits of $\mathrm{O}_{2m}^\varepsilon(q)$ on $V \setminus \{0\}$, and (6) is now proved.

Since π^+ is multiplicity-free, up to reordering, by (3) and (6), we may assume that

$$\pi^+ = 1 + a\chi^- + b\chi^+ + \sum_{i=1}^{t} \chi_i + \rho,$$

where $a, b \in \{0, 1\}$, $0 \leq t \leq \frac{q}{2} - 1$ and $\langle \pi_0, \rho \rangle = 0$. By (6), we have $q - 2 \geq 2t = q - 1 - a - b \geq q - 3$. Hence $2t = q - 2$ and $\{a, b\} = \{0, 1\}$. Since $\pi^+(1) = |\Omega^+| = q^m(q^m + 1)/2$ and $\pi'(1) = (q/2 - 1)|G : P| = (q/2 - 1)(q^{2m} - 1)/(q - 1)$, it follows by (2) that $a = 0$ and $b = 1$. By (4), we have $\pi^+ = 1 + \chi^+ + \pi'$.

Now (3), (4), (5) and (6) imply immediately that $\pi^- = 1 + \chi^- + \pi'$. This shows that

$$\pi^+ + \pi^- = (1 + \chi^+ + \pi') + (1 + \chi^- + \pi') = 1 + 1 + \chi^+ + \chi^- + 2\pi' = 1 + \pi_0 = \pi,$$

which completes the proof of Theorem 1. □

We note that the "$q = 2$" case of Theorem 1 was first proved by Siemons and Zalesskii in [13, Proposition 3.1]. This case is particularly easy to deal with (considering that the action of G on both Ω^+ and Ω^- is 2-transitive) and its proof depends only on Frobenius reciprocity. However, the general statement (valid for every even q) of Theorem 1 was undoubtedly inspired by their observation.

Theorem 1 reproduces the following result as an immediate corollary (see [4, Theorem 6]).

Corollary 2. *Every element of* $\mathrm{Sp}_{2m}(q)$ *is conjugate to an element of* $\mathrm{O}_{2m}^+(q)$ *or of* $\mathrm{O}_{2m}^-(q)$.

Proof. Let $g \in \mathrm{Sp}_{2m}(q)$. Then $\pi(g) = (1 + \pi_0)(g) = 1(g) + \pi_0(g) \geq 1(g) = 1$ and therefore, since $\pi = \pi^+ + \pi^-$ by Theorem 1, either $\pi^+(g) \geq 1$ or $\pi^-(g) \geq 1$; that is, g fixes some point in Ω^+ or in Ω^-. In the first case g has a conjugate in $\mathrm{O}_{2m}^+(q)$ and in the second case g has a conjugate in $\mathrm{O}_{2m}^-(q)$. □

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Simon Guest, Mathematics, University of Southampton, Highfield, SO17 1BJ, United Kingdom
E-mail address: `s.d.guest@soton.ac.uk`

Dipartimento di Matematica e Applicazioni, University of Milano-Bicocca, Via Cozzi 53, 20125 Milano, Italy
E-mail address: `andrea.previtali@unimib.it, pablo.spiga@unimib.it`